SECTION II.—GENERAL METEOROLOGY.

REPORT ON MODES OF AIR MOTION AND THE EQUATIONS OF THE GENERAL CIRCULATION OF THE EARTH'S ATMOSPHERE.

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The object of the present investigation is to classify and to analyze fundamental modes of air motion and to ascertain some of the mathematical laws governing general circulation, a particular mode of motion in the earth's atmosphere.

The problems under consideration had their origin in the experiments of Osbourne Reynolds (**19–22**),[1] who demonstrated that the mean drift of a liquid in turbulent motion does not obey the laws expressed by the standard hydrodynamical equations. These equations are derived for the detailed motion of fluid elements under the assumption that the velocities and their space derivatives are so small that their powers and products may be neglected (**73**, p. 554). Reynolds obtained modified hydrodynamical equations expressing the average drift of the liquid, not in terms of the turbulent motion,[2] but in terms of mean values of functions of the deviations of the velocities of the fluid elements from the velocity of the average drift. These equations are open to the serious objection that they are obtained by applying the standard hydrodynamical equations to the detailed turbulent motions of the fluid elements, which motions in practice do not satisfy the conditions just stated upon which the validity of the hydrodynamical equations depends.

Clerk Maxwell (**91**, p. 81) demonstrated that, under certain conditions, the classical hydrodynamical equations expressed the laws governing the mass motion of a gas. Recent experiments [3] with the flow of gases in wide tubes and empirical studies of surface wind data [4] have shown that these equations not only fail to express the laws governing the turbulent flow of liquids but they also fail to express the laws governing the turbulent drift of the winds.

The reason for this apparent failure of the Kinetic Theory of Gases to account for observed air motions is to be found in the fact that the classical equations apply to the instantaneous air drift out of one fixed, microscopic volume element into another, while the motions that actually come under observation are in general of another and fundamentally different mode.

With a view to removing this difficulty, air motions are classified in the following pages in terms of mass motion, molecular motion, winds, and turbulence. Winds are classified with reference to their order, and turbulent motions, with reference to their order and to their kind.

Mathematical expressions for mass motion and for molecular motion are already available from the standard Kinetic Theory of Gases, but, strange as it may seem, no attempt appears to have yet been made to formulate mathematical expressions descriptive of *winds* and of *turbulent motion*. In other words, no definitions of these fundamentally important modes of motion have been proposed which possess a degree of precision sufficient for the application of the methods of Mathematical Physics to the study of the circulation of the atmosphere.

Winds have accordingly been defined with precision in the present article in terms of multiple mean-value integrals. These integrals are built up from certain fundamental concepts of classical gas theory. The idea of wind order is suggested immediately by this definition, and the quantitative definitions of turbulence and of orders of turbulence grow out of the integral expressions for winds and the idea of wind order.

These definitions, on the one hand, are expressed in terms of Mathematical Analysis. On the other hand, they describe with precision observed phenomena as yielded by instrumental observations.

Having set up these analytical definitions of fundamental modes of air motion and established their correspondence to observed phenomena, it is here shown that equations for the winds of the general circulation follow as an extension of the classical Kinetic Theory of Gases.

Mass motion.—Confining the present investigation to the motions of the earth's atmosphere below about 80 kilometers elevation (**101**, pp. 1–29), we have to deal with an aggregation of molecules, chiefly of two kinds—nitrogen and oxygen. According to the Kinetic Theory of Gases, each one of these molecules describes a rectilinear free path with uniform velocity, until it encounters one or more neighboring molecules, when it starts off in a new direction at a different speed. For example, at 0° C. and under a pressure of one atmosphere the number of molecules in one cubic centimeter of air is about 2.75×10^{19}. The mean length of the free paths is about 1.42×10^{-5} cm. At 15° C., the mean velocity of the molecules is about 459 meters per second, and the range of velocities represented at any instant by the individual molecules is very great. The number of encounters per second in one cubic centimeter is of the order of 1.64×10^{29}.

To investigate the motion of this complicated dynamical system, it is usual to regard the space occupied by the air as divided into a great number of cells. These cells are fixed in space and are small compared with the total volume occupied by the gas. A typical cell is chosen as a region for investigation. At a given instant the velocity of each of the swarm of molecules then occupying the cell is supposed to be ascertained. The numerical mean of all these velocities is then computed, and the result is a measure of the tendency of the molecules to drift collectively out of one cell into another. In other words, the numerical mean measures the *mass motion* of the gas in the cell.

Evidently the character of the mass motion corresponding to a given state of motion of the molecules depends upon the size arbitrarily assigned to the fixed cells. If we suppose the cells all alike, and denote their common volume by $d\tau$, then $d\tau$ should satisfy the following conditions:

(*a*) It should be large compared with the mean free path of the molecules.

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[1] NOTE.—References are denoted by their serial number in the bibliography at the end of this paper.

[2] See definition, p. 315.

[3] See references cited relative to kinetic turbulence, p. 315.

[4] 70, 84, 71, 72, 94, 95, 99.

(*b*) It should be so small that the molecular density is practically uniform throughout each cell.

The position of any of one these microscopic cells, or volume elements, may be indicated by the coordinates of its middle point P. We shall refer to the mass motion in this cell as the *mass motion at point P*.

The temperature and the hydrostatic pressure of classical gas theory are essentially numerical means computed for the molecules occupying a specified cell at a given time; and molecular density is explicitly defined as the number of molecules in the cell under consideration, divided by its volume. The temperature, the hydrostatic pressure, and the density of the atmosphere may therefore be regarded as phases of mass motion.

Integral expressions for mass motion.[5]—In terms of Mathematical Analysis, the mass motion of a gas can be described with precision. Let x, y, and z be the coordinates, referred to fixed axes, of a point within a fixed volume element $d\tau$, and let $\nu\ (x, y, z, t)$ be the molecular density in this element at a given instant t. Imagine the velocity diagram of the Kinetic Theory of Gases constructed for all the molecules in $d\tau$ at the instant t. The molecular velocities in $d\tau$ are supposed to vary from molecule to molecule through all possible values. It is usual in works on the Kinetic Theory of Gases to select from the molecules in $d\tau$ at instant t all those whose velocities lie in a range

$$\left.\begin{array}{l} u \text{ and } u+du \\ v \text{ and } v+dv \\ w \text{ and } w+dw \end{array}\right\} \text{- - - - - - - - - - - - - - - -(A)}$$

which molecules we shall designate as molecules of class A, and to define a function $f(u, v, w, x, y, z, t)$ as such that the number of molecules of class A in $d\tau$, at instant t, is

$$\nu\ (x, y, z, t)\ f\ (u, v, w, x, y, z, t)\ d\varpi d\tau \text{- - - - - -(1)}$$

where

$$d\varpi = dudvdw$$

and is the volume element corresponding to class A in the velocity diagram. If u, v, and w are the velocity components of an individual molecule in a cell whose center is $P\ (x, y, z)$, and if $\varphi\ (u, v, w)$ is a function depending on them,[6] then, with the aid of the foregoing definition, the numerical mean of φ for all the molecules in the volume element $d\tau$ may be expressed by the relation

$$\varphi_0 = \frac{1}{\nu d\tau} \Sigma \varphi \nu f d\varpi d\tau$$

where the summation is effected for the velocities of all the molecules in $d\tau$. Approximately,

$$\varphi_0 = \frac{1}{\nu d\tau} \int_{-\infty}^{+\infty} \varphi \nu f d\varpi d\tau = \int_{-\infty}^{+\infty} \varphi f d\varpi \text{- - - - - - - - - - - (2)}$$

Thus, for example, the x-component of the velocity of mass motion is

$$u_0 = \int_{-\infty}^{+\infty} u f d\varpi$$

A numerical mean is in general a function of the time.

Molecular motion.—Deviations of the motions of individual molecules from the mass motion give rise to another mode of motion. Let u, v and w be the velocity components of an individual molecule in a fixed volume element $d\tau$ at the instant t, and let u_0, v_0, and w_0 be the numerical means of the corresponding velocity components of all the molecules in that cell at the same instant; then the velocity whose components are U, V and W which is defined by the relations

$$u = u_0 + U,\ v = v_0 + V,\ w = w_0 + W \text{- - - - - - - -(3)}$$

is called the *molecular velocity* of the molecule under consideration.

Mean-value properties of mass motion.—If φ of equation (2) is now set successively equal to u, v, and w, then the resulting numerical means are equal respectively to the velocity components of the mass motion of the gas. Considerable information regarding these means is available from the Kinetic Theory of Gases. For example, in the integral

$$(\varphi_0)_0 = \int_{-\infty}^{+\infty} \varphi_0 f d\varpi \text{- - - - - - - - - - - - - - - -(4)}$$

the summation is effected for all the molecules in the volume element $d\tau$. But φ_0 is the same for all the molecules in $d\tau$, hence[7]

$$(\varphi_0)_0 = \varphi_0 \int_{-\infty}^{+\infty} f d\varpi = \varphi_0 \text{- - - - - - - - - - - - - - (5)}$$

From this relation and (3), we have

$$U_0 = V_0 = W_0 = O \text{- - - - - - - - - - - - - (6)}$$

consequently

$$\left.\begin{array}{l} (u^2)_0 = u^2_0 + (U^2)_0 \\ (v^2)_0 = v^2_0 + (V^2)_0 \\ (w^2)_0 = w^2_0 + (W^2)_0 \end{array}\right\} \text{- - - - - - - - - - - - - (7)}$$

and

$$\left.\begin{array}{l} (uv)_0 = u_0 v_0 + (UV)_0 \\ (vw)_0 = v_0 w_0 + (VW)_0 \\ (wu)_0 = w_0 u_0 + (WU)_0 \end{array}\right\} \text{- - - - - - - - - - - - (8)}$$

Assuming, as a first approximation, that the law of equipartition of energy

$$(U^2)_0 = (V^2)_0 = (W^2)_0 \text{- - - - - - - - - - - - - (9)}$$

holds for the molecules which at any instant are in a given volume element $d\tau$, it can be shown that in $d\tau$,

$$\left.\begin{array}{l} (UV)_0 = -\frac{\kappa}{\rho}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right) \\ (VW)_0 = -\frac{\kappa}{\rho}\left(\frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial z}\right) \\ (WU)_0 = -\frac{\kappa}{\rho}\left(\frac{\partial u_0}{\partial z} + \frac{\partial w_0}{\partial x}\right) \end{array}\right\} \text{- - - - - - - - - - (10)}$$

where κ corresponds to the viscosity coefficient of the Theory of Continuous Media, and ρ is the volume density of the gas.

It should be observed that the assumption of the law of equipartition of energy applies here to deviations from

[5] For the sake of brevity, the expressions here given apply to a single gas. They may, however, be extended to apply to a mixture of gases having different molecular weights.

[6] For example, u^2, or uv or uv^2.

[7] The total number of molecules in $d\tau$ is $\nu d\tau \int_{-\infty}^{+\infty} f d\varpi$; but the total number is also given by $\nu d\tau$; hence $\int_{-\infty}^{+\infty} f d\varpi = 1$.

numerical averages taken over any one of the very small volume elements under consideration, and not to deviations from numerical averages taken over large masses of the atmosphere.

The hydrostatic pressure p of the gas is defined as

$$p=\frac{\rho}{3}(U^2+V^2+W^2)_o \text{ . . . . } (11)$$

and the temperature T is given by the equation

$$T=\frac{1}{3H}(U^2+V^2+W^2)_o \text{ . . . . } (12)$$

in which H is the constant of the characteristic equation of the gas.

The equations of mass motion.—In the Kinetic Theory of Gases, it is also shown that by means of a study of the flux of matter, of momentum, and of kinetic energy, as governed by the fundamental laws of dynamics, a system of equations may be deduced for the mass motion of a gas. The results depend on the foregoing mean value properties of mass motion.

The dynamical equations.[8]—The analytical statement of the law of conservation of matter takes the form of the *equation of continuity:*

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_o)}{\partial x}+\frac{\partial(\rho v_o)}{\partial y}+\frac{\partial(\rho w_o)}{\partial z}=O \text{ . . . . } (13)$$

and the law of conservation of momentum is expressed by the three *momentum equations:*[9]

$$\left.\begin{aligned}\frac{du_o}{dt}&=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\epsilon}{3}\frac{\partial\theta_o}{\partial x}+\epsilon\Delta u_o\\ \frac{dv_o}{dt}&=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\epsilon}{3}\frac{\partial\theta_o}{\partial y}+\epsilon\Delta v_o\\ \frac{dw_o}{dt}&=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\epsilon}{3}\frac{\partial\theta_o}{\partial z}+\epsilon\Delta w_o\end{aligned}\right\}\text{ . . . . } (14)$$

The thermodynamical equations.—The *characteristic equation* of the gas is

$$p=\rho HT \text{ . . . . } (15)$$

Finally, the law of conservation of energy yields the *energy equation:*

$$\frac{\vartheta}{\rho}\Delta T+\frac{1}{\rho}\mathfrak{G}=c_v\frac{dT}{dt}+\tfrac{2}{3}c_v\theta_o+\frac{\epsilon}{J}W \text{ . . . . . . } (16).$$

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[8] First derived from gas theory considerations by Maxwell, see 91, p. 81.

[9] *Notation.*—(X, Y, Z) is the resultant of the body forces per unit mass.
T is the temperature, measured on the absolute scale.

$\frac{d}{dt}=\frac{\partial}{\partial t}+u_o\frac{\partial}{\partial x}+v_o\frac{\partial}{\partial y}+w_o\frac{\partial}{\partial z}.$

$\theta_o=\frac{\partial u_o}{\partial x}+\frac{\partial v_o}{\partial y}+\frac{\partial w_o}{\partial z}$, which is analogous to the time rate of cubical dilitation in the case of the motion of a continuous medium.

$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, the Laplacian operator.

κ is analogous to the viscosity coefficient of a continuous medium.
ϑ is the coefficient of conduction of heat for the gas.
c_v is the specific heat of the gas at constant volume.
H is the gas constant in the characteristic equation of the gas.
J is the mechanical equivalent of heat.

$\epsilon=\frac{\kappa}{\rho}.$

$\mathfrak{G}$ is the gain in energy per unit volume due to absorption and emission of radiation.

$$W=\frac{2}{3}\theta_o^2-\left[\left(\frac{\partial v_o}{\partial x}+\frac{\partial u_o}{\partial y}\right)^2+\left(\frac{\partial w_o}{\partial y}+\frac{\partial v_o}{\partial z}\right)^2+\left(\frac{\partial u_o}{\partial z}+\frac{\partial w_o}{\partial x}\right)^2\right]$$
$$-2\left[\left(\frac{\partial u_o}{\partial x}\right)^2+\left(\frac{\partial v_o}{\partial y}\right)^2+\left(\frac{\partial w_o}{\partial z}\right)^2\right],$$

which is twice Stokes' dissipative function for a continuous medium.

Space derivatives of the temperature and of the velocities.—Very little is now known regarding the magnitudes of the space derivatives of the temperature and of the velocities of the mass motion of the earth's atmosphere, as compared with the magnitudes of the "gradient" and body forces[10] which give rise to that motion. This much, however, seems certain, that for mass motion in the neighborhood of an obstacle or near the ground where cross currents and eddies prevail, these derivatives may be relatively large, and hence by no means negligible for purposes of approximation.

Mass motion and observed phenomena.—The validity of equations (14), in so far as they express the laws governing lamellar (nonturbulent) gas motions, has been abundantly established by experimentation.[11] At the same time, observation and experiment have demonstrated that they fail to express the laws governing those turbulent, eddying, tumultuous motions characteristic of the circulation of the atmosphere.[12]

Notwithstanding this notable failure, the mathematical investigation of air motions had not been extended beyond the foregoing classical analysis of mass motion until, through the researches leading to the present report, it became evident that the apparent breakdown of the equations of mass motion was due to the fact that, on the one hand, the mass motion of the atmosphere can not generally be observed by ordinary methods, and, on the other hand, the modes of motion actually observed and recorded by the Government observers and others are not mass motions but *time means of mass motions, averaged over intervals ranging from a few seconds to a long period of years.*

Evidence of this fundamental and important fact will be brought forward in the course of the ensuing discussion of winds and turbulence. Meanwhile, we proceed to the following analytical definition of a wind.

Definition of a wind.—Let φ_o be any one of the mass motion variables u_o, v_o, w_o, p, ρ, or T, and let $\mathfrak{T}$ be a fixed, definite interval of time. Then the time mean of φ_o extended over the interval $\mathfrak{T}$ is given by

$$\bar{\varphi}_o=\frac{1}{\mathfrak{T}}\int_{t-\mathfrak{T}/2}^{t+\mathfrak{T}/2}\varphi_o dt \text{ . . . . } (17)$$

If φ_o is one of the velocity components u_o, v_o, or w_o, then the time mean takes the form of the double mean-value integral

$$\bar{\varphi}_o=\frac{1}{\mathfrak{T}}\int_{t-\mathfrak{T}/2}^{t+\mathfrak{T}/2}\int_{-\infty}^{+\infty}\varphi f d\varpi dt \text{ . . . . } (18)$$

The collective mean-value phenomena of velocity, pressure, temperature, and density, as defined by relations (17) and (18), will be said to constitute a *wind of order* $\mathfrak{T}$.

Observed phenomena and the integral expressions for a wind.—The classical Kinetic Theory of Gases, as we have seen, treats of mass motion and the molecular motion accompanying it, and it has in general been tacitly assumed that observations made in the ordinary way with standard meteorological instruments represent the mass motion of the air. But this assumption clearly does not accord with the facts. An ideal anemometer, for

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[10] The "gradient" force is the force per unit mass whose components are given by the pressure terms of equations (14). The body forces (X, Y, Z) of equations (14) are the forces per unit mass due to gravitational and electromagnetic fields existing in the region occupied by the atmosphere.

[11] See, for example, 62, 63, 64, 65, 66, 67, 68, 69, 96.

[12] 70, 71, 72, 6, 12.

example, so minute as to be contained in one of the elementary cells $d\tau$ (see p. 311), so designed as to set up no extraneous disturbances, and so sensitive as to respond instantly to the most rapid or minute fluctuations of the air, would measure the air drift in the cell $d\tau$, that is to say, the velocity of the mass motion as defined by equation (2).

The standard Robinson anemometer occupies a relatively large space, it sets up eddies and cross currents in its neighborhood, it fails to respond to rapid or minute fluctuations of the air (**1**, **100**, **61**), and its moment of inertia is so large that, on gusty days, it constantly lags or overruns.

Only approximately, therefore, does the frequency of revolution of the whirling cups indicate the true mass motion which, if the anemometer were removed, the air would have in the cell $d\tau$ at the center of the instrument.

In practice, however, the detailed frequency variations of the instrument are not ascertained. The data are obtained by counting the number of notches appearing per specified time upon the record sheet.

Subject, then, to errors due to (*a*) the large size of the instrument, (*b*) the disturbances which it sets up, (*c*) its failure to respond to rapid and minute fluctuations, and (*d*) its tendency to lag during sudden gusts and to overrun during sudden calms, actual readings of the anemometer indicate for specified values of $\mathfrak{T}$, time means of the mass motion of the air in the neighborhood of the instrument, in the sense denoted by the multiple mean value integral (18).

When the air is in motion, the conditions are very much the same in the case of the standard thermometer. An ideal, microscopic thermometer, suspended in a cell $d\tau$, creating no extraneous disturbances, and quick and sensitive enough to yield readings proportional to the most rapid and minute variations of the mean kinetic energy of the molecular motion in that cell, would register the temperature of the air as defined by equation (12).

The large size of the standard thermometer compared with $d\tau$, the eddies it sets up, its sluggish reactions and its inability to respond to minute temperature variations (**2**) must greatly impair the accuracy with which the variations of the instrument indicate the actual fluctuations of the temperature.

Unlike the anemometer, the thermometer yields, from a single reading, not a definite time mean, but a time mean corresponding to some indeterminate and unspecified value of $\mathfrak{T}$, varying from reading to reading, and depending partly on the reaction constants of the instrument, and partly on the observer.

When mean temperatures are obtained by averaging such readings over fixed, definite, and specified values of $\mathfrak{T}$ (daily means, annual means, etc.) then, subject to the errors described, they yield the wind temperatures defined by the integral expression (17).

It is essential to observe that instrumental errors in wind velocity and temperature, due to the causes enumerated, pertain to *the mass motion of classical gas theory* and not to the time means discussed in this paper. The latter can be computed with precision.[13]

The dependence of winds corresponding to a given mass motion upon the values assigned to $\mathfrak{T}$.—Fifteen readings of a Robinson anemometer, at Madison, Wis., on August 26, 1897, are represented in figure 1. The velocities having been obtained by counting the number of notches appearing per hour on the record sheet, the interval $\mathfrak{T}$ for these observations was one hour. Figure 2 shows the wind velocity at the same station, but averaged for each

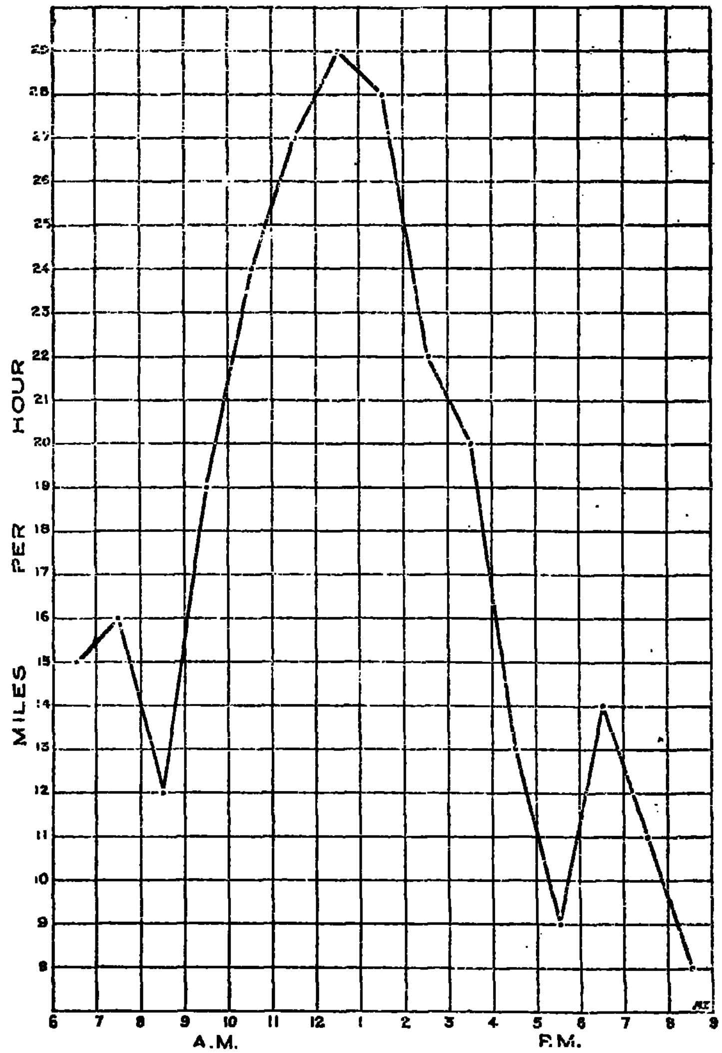

FIG. 1.—Wind velocity at Madison, Wis., August 26, 1897. $\mathfrak{T}$=1 hour.

month of the year 1897, twelve mean values for which $\mathfrak{T}$ was one month. The yearly mean wind velocities for the years 1894 to 1903, inclusive, are shown in figure 3. In

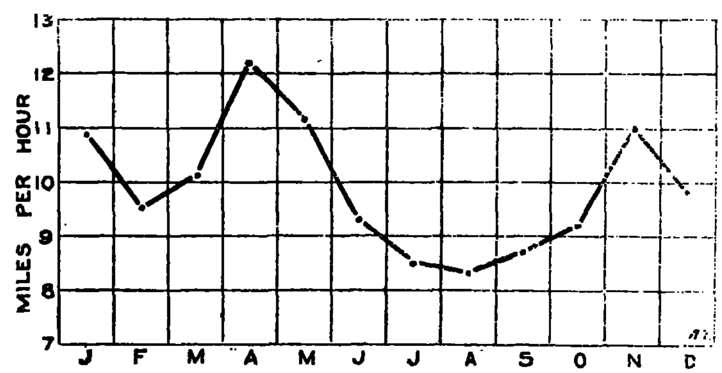

FIG. 2.—Wind velocity at Madison, Wis., during 1897. $\mathfrak{T}$=1 month.

this case $\mathfrak{T}$ was one year, each mean having been computed from data obtained during the interval between January 1 and December 31 of the year under consideration. Ten mean values are plotted in figure 4, for which

[13] The dynamical significance of the barometer is briefly discussed in footnote to page 321.

$\mathfrak{T}$ was thirty years; that is to say, the mean wind velocity w as found from anemometer readings extending fifteen

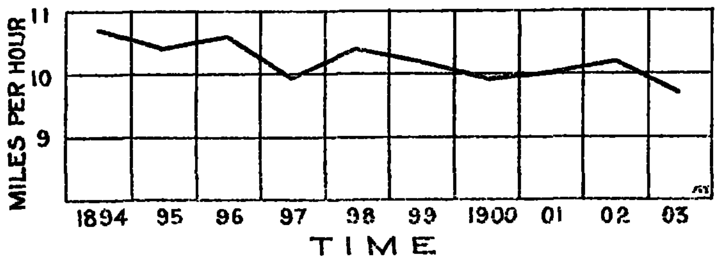

FIG. 3.—Wind velocity at Madison, Wis., 1894–1903. $\mathfrak{T}$=1 year.

years before and after the year for which the mean was computed.[14]

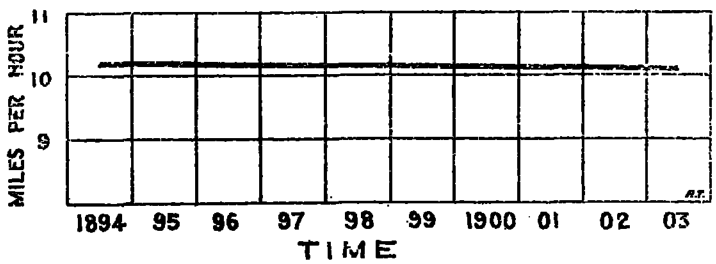

FIG. 4.—Wind velocity at Madison, Wis., 1894–1903. $\mathfrak{T}$=30 years.

For purposes of comparison, these four graphs should be drawn to the same scale. Thus, for example, if variations of the type shown in figure 1 were plotted to the scale of figures 3 and 4 over the entire period of ten years, the resulting graph would have the appearance of *a continuous band nearly seventeen units broad.* The extreme mutual divergence of these winds is apparent.

Unfortunately, wind direction data, as now recorded, do not afford material for the construction of stream-line charts for *winds* in the sense in which that term is here used. At the same time, it is unquestionably true that stream lines, like velocities, exhibit a fundamental difference in the character of the winds corresponding to the same mass motion but to different values of $\mathfrak{T}$.

The same phenomenon is observable in atmospheric temperatures and pressures (2).

It may therefore be stated as a general principle, that the character of a wind corresponding to a given mass motion is, in general, profoundly affected by the value arbitrarily assigned to the time interval $\mathfrak{T}$.

From this principle it follows that no specific problem in wind motion can be regarded as formulated, unless the order of the wind under consideration be explicitly defined.

Turbulent motion.—It will be observed that in taking the monthly averages for figure 2, the characteristic irregularities of the wind velocities of figure 1 are largely smoothed out; and that in taking the yearly averages for figure 3, not only the characteristic irregularities of the wind velocities of figure 1, but those of figure 2 are smoothed out also. The averaging process corresponding to figure 4, for which $\mathfrak{T}$ was thirty years, had the effect of practically obliterating all irregularities, so that the graph appears as a straight line nearly parallel to the axis of abscissas.[15] The situation is in general the same relative to pressure curves, stream lines, etc. There are cases, however, where the mean motion does not change with $\mathfrak{T}$. If, for example, the wick of a kerosene lamp be adjusted to give a steady flame without smoking, stream lines in the chimney for five-minute averages differ very little from stream lines for five-second averages. But when the motions differ for different values of $\mathfrak{T}$, it is essential to distinguish clearly between them. This can be accomplished analytically by means of the two following definitions.

(*a*) A wind of order $\mathfrak{T}$ may be described in terms of *time curves* representing the velocity, pressure, temperature and density at a single station during a given period of time; or else by a *weather map* showing stream lines, isobars, isotherms and lines of constant density drawn over a given region for a single instant of time.

(*b*) Given an air mass, the molecules of which are in a given state of motion M. Suppose that a $\left\{\begin{matrix}\text{set of time curves}\\ \text{weather map}\end{matrix}\right\}$ S be constructed representing the motion M for time means extended over an interval $\mathfrak{T}$. If another $\left\{\begin{matrix}\text{set of time curves}\\ \text{weather map}\end{matrix}\right\}$ S_1 be constructed representing the motion M, but for time means extended over a shorter interval $\mathfrak{T}_1$, then, provided the $\left\{\begin{matrix}\text{curves}\\ \text{map}\end{matrix}\right\}$ S_1 be different from the $\left\{\begin{matrix}\text{curves}\\ \text{map}\end{matrix}\right\}$ S, the motion S_1 will be said to be *turbulent relative to the motion S.*

Turbulence of different orders.—The order n of the turbulent motion S_1 relative to S will be defined as

$$n=\frac{\mathfrak{T}}{\mathfrak{T}_1} \qquad \text{(18a)}$$

Figure 2, for example, shows turbulence of order 12 relative to the wind defined for annual means. Figure 1 shows turbulence of order 8, 760 relative to the wind defined for annual means, and of order 262,800 relative to the wind defined for 30 year means.

Kinetic turbulence.—Let us return to the illustration of the kerosene lamp. By properly adjusting the flame, stream lines for half second averages can be made smooth; in which case the draft is good, and the lamp burns without smoking. Turn the wick up, and the draft, or *wind*, is accelerated, until what seems a critical velocity is attained, at which the stream lines become unstable, winding themselves into a tangle revealed by spirally curling smoke jets due to imperfect combustion. The imperfect combustion is in turn due to decreased draft, some of the kinetic wind energy having gone into the production of turbulence.

A great deal of experimental work has been done on turbulence of this kind, in both liquids and gases. Some of the testimony relative to critical velocities is conflicting, and the character of the stream lines and the origin of the motion itself are little understood.[16]

The first researches in kinetic turbulence seem to have been made in 1883 by Osbourne Reynolds (19–22), whose notable experiments with liquids revealed clearly the complicated character of the secondary stream lines often associated with a general drift or mean motion. For slow motions, the stream lines indicated the well-known Poisuille *régime* (18); but mean motion accelerations led to more or less abrupt appearances of exceedingly complex stream-line configurations which, super-

[14] See formula (18).
[15] The apparent slope of the curve of fig. 4 is discussed on p. 317, note 24.
[16] 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12; 13, No. 3; 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26; 93. p. 776; 94, 95, 99, 103.

posed on the general drift, absorbed much of its kinetic energy.

Much stress was laid by Reynolds on the abruptness of the transition from the Poisuille *régime* to turbulent motion, but more recent researches, in particular those of Ekman (**5**) in 1907, seem to indicate that those changes, occurring at certain "critical velocities," were due to fortuitous disturbances set up by the apparatus. This, however, is of secondary importance. What Reynolds demonstrated was that, except in the case of slow motions, the general drift defined by time means was accompanied by superimposed turbulence, and that the former was profoundly affected by the latter. From the point of view of the present paper, Reynolds's turbulence was a mean motion corresponding to a value of $\mathfrak{T}$ considerably less than the order of the general drift.

In gases, related phenomena have been studied experimentally by Becker (1907), Fry and Tyndall (1911), Dowling (1912), Kohlrausch (1914), Zemplén (1912), and others. The experiments of Kohlrausch (**12**) are of particular interest, as showing clearly a deep-seated retarding influence at work after the transition from the Poisuille to the turbulent *régime*.

A large proportion of experimentation in air turbulence has been confined to the investigation of critical velocities, several of which have sometimes been observed in connection with a single type of apparatus. Zemplén's (**25**, p. 71) experiments with spherical gas layers have, however, indicated that the abrupt appearance of turbulence upon the attainment of certain critical velocities was probably due to disturbances set up by the ends of the tubes through which the gas was driven.

Varied and in some respects conflicting as the data seem to be, the following generalizations seem to have been empirically established. (*a*) Turbulence of the kind under consideration appears in the neighborhood of obstacles, material boundaries and surfaces of discontinuity, and in general where high differential velocities exist between adjacent fluid strata. (*b*) It may appear where the material boundaries and the adjacent fluid strata are at the same temperature. (*c*) A wind of order $\mathfrak{T}$ is in general modified profoundly by the occurrence of superimposed gusts, cross currents, and eddies—turbulent motions relative to $\mathfrak{T}$. In particular, a turbulent wind does not obey the laws expressed by equations (13) to (16) governing the mass motion of the gas.[17] On account of the first two of the foregoing generalizations, the kind of turbulence under consideration will be referred to as *kinetic turbulence*.[18]

The results stated in the last paragraph have been derived from observation and experiment. Reynolds and Lorentz (**19-22**), (**13**) have shown from theoretical considerations why the hydrodynamical equations do not govern the mean motion, or average drift, of a turbulent liquid. In the present paper it is shown from theoretical considerations why the equations of mass motion (13) to (16) do not govern a turbulent wind, defined for large values of $\mathfrak{T}$.

Convective turbulence.—A stream of water at 20°C. and flowing rapidly down an inclined iron trough at the same temperature, will exhibit turbulence relative to a general drift corresponding, say, to hourly means. If the temperature of the trough be sufficiently increased, the water will boil as it descends, and the turbulence will thus be enormously complicated by the appearance of a new type of stream-line irregularities. Such irregularities, which are due to temperature differences between material boundaries and their adjacent fluid strata, occur in the lower layers of the atmosphere and operate powerfully to modify atmospheric stream lines. On a fair summer day the process is generally visible. During the morning, the earth becomes hotter than the adjacent air, and heat is conducted from the ground to the lowest air stratum. The resulting vertical instability manifests itself in multitudinous flickering jets rising from the ground. Although these jets are visible only to the height of a few feet, the appearance of cumulous clouds bears witness to the fact that the jets are uniting into massive convection columns which are driven upward to an altitude of perhaps a mile. The structure of higher order surface winds is profoundly complicated by these enormous air columns cutting across their lines of flow. Relatively lacking in momentum, the ascending masses mix with the currents of the upper air, while irregular downward draughts, replacing the thermally driven masses and transmitting horizontal momentum from the upper air currents to the surface layers, unite into a wind of higher order near the ground, which, making itself first felt in the morning, attains a maximum velocity during the day, and disappears during the late hours of the afternoon.[19]

This mode of motion, *convective turbulence*, is characteristic of regions where the material boundaries and the adjacent air strata are at different temperatures. It accordingly differs essentially from kinetic turbulence with regard to its origin; possibly also with regard to characteristic stream-line configurations.

Combined effects of kinetic and convective turbulence.—Fair weather convection, as we have seen, affords an illustration of convective turbulence relative, for example, to a wind defined for daily averages. Local thunderstorms furnish another example. Both, of course, are complicated by the action of water vapor. The HIGHS and LOWS of the daily weather maps reveal turbulence relative to monthly and yearly means. Some of these disturbances, like the Arizona LOW of the summer months, belong clearly, at least in their incipient stages, to the convective class. Others seem to be purely kinetic in their origin. The fitful and intermittent gusts of the substratosphere[20] are doubtless of the same character. Probably most of the turbulence phenomena (relative to yearly means) depicted on the daily weather maps are resultants of both kinds of turbulent motion. In fact it is by no means possible in atmospheric phenomena near the ground to distinguish between irregularities of the two sorts. Observational studies of turbulence in the lower layers of the atmosphere unquestionably pertain to a mixture of the two. Lilienthal's (**61**, p. 77) studies carried on with vertical recording systems of parallel wind vanes revealed marked turbulence in winds of order ranging from three to six seconds. It was found that winds in passing over a wood or a hill sent great eddies upward. The pronounced and almost abrupt upward motions usually seen to accompany the mean drift of smoke jets from tall, isolated factory chimneys were shown to be due, not to the heated condition of the escaping gases, but to characteristic upward irregularities of wind motion near the ground. Similar phenomena were observed and studied by Langley (**1**), whose measurements with sensitive, light-weight anemometers revealed marked turbulence in winds defined

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[17] See **19–22, 13, 8, 71, 72,** and standard works dealing with the flow of fluids through pipes, weirs, canals, etc.

[18] Regions of turbulence in the atmosphere are sometimes bounded by surfaces along which the wind is discontinuous. Some of the more important memoirs relating to surfaces of discontinuity are the following, arranged here in chronological order: **53**, p. 310; **52, 39, 46, 50, 55**: **43**, p. 146: **28, 40, 41, 42, 38, 48, 45, 27, 49, 29, 47, 30, 54, 31, 32, 33, 34, 44, 35, 37, 36, 51, 89, 104**. Reference **92**, p. 107, is of interest in connection with **45**. An admirable summary and extensions of the theory are contained in **98**.

[19] See **102** and **99** on the diurnal variation of the wind.

[20] **56**, p. 165; **57**, p. 214; **58**, p. 153; **59**, p. 26; **60**.

for $\mathfrak{T}$-intervals of from four to ten minutes. Barkow (2; see also 90) showed that turbulence of the same order reveals itself in temperature and pressure graphs. But in the results of observation there is generally little or nothing to enable one to distinguish between kinetic and convective turbulence. Near the earth the two phenomena combine to the extent that the atmosphere has well been called a "treacherous sea," where irregular cross currents and eddies, violent swirls and sudden side gusts prevail.[21]

That the two kinds of turbulence are, however, essentially different seems to be indicated by the form of the equations of general circulation (45) to (50).

Mean value properties of winds.—Let α represent any one of the six mass motion variables u_0, v_0, w_0, p, ρ and T. Deviations of the mass motion α [22] in any volume element $d\tau$ from a wind due to α and defined in terms of an interval $\mathfrak{T}$ [23] are given by the relation

$$\alpha = \overline{\alpha} + \alpha' \qquad (19)$$

As was pointed out in the discussion of turbulent motion, *the larger the value of $\mathfrak{T}$, the more nearly the corresponding wind approximates a state defined by the relation*

$$\frac{\partial \overline{\alpha}}{\partial t} = 0 \qquad (20)$$

This general principle is assumed here as a fundamental hypothesis. It is well illustrated by the wind velocities represented in figures 1 to 4. The great irregularities shown in figures 1, 2, and 3 disappear in figure 4, which is in fact a straight line, approximately parallel to the time axis.[24] Figure 4 therefore indicates that the mean velocity for 30-year averages satisfies relation (20) throughout the 10-year interval represented.

This relation is a sufficient condition for a steady state of the mean motion $\overline{\alpha}$. But this is by no means tantamount to saying that the wind thus defined is a *steady wind*, as that term is ordinarily used. Quite the contrary, winds of 30-year order are in general turbulent. Shorter $\mathfrak{T}$-intervals for the same mass motion, as illustrated in figures 1, 2, and 3, yield vortices and cross currents due to obstacles, convectional swirls and ascending air columns, cyclonic storms and anticyclones and a host of other irregularities, all unsteady in the extreme.

Proceeding therefore from the fundamental hypothesis (20) one finds that, *for sufficiently long values of $\mathfrak{T}$,*

$$\overline{\overline{\alpha}} = \overline{\alpha} \qquad (21)$$

hence by (19)

$$\overline{\alpha'} = 0 \qquad (22)$$

[21] Rotch and Palmer (100, p. 44) observed, in a 33-mile wind, velocity fluctuations which were between 15 and 75 miles per hour, "Including a succession of minor oscillations, some lasting only a few seconds. As many as ten pulsations in pressure per second have been noted in these rapid gusts."

[22] See equation (2).

[23] See equations (17) and (18).

[24] The apparent velocity decrease in fig. 4 of a little over 1 per cent, during the 10-year interval, was doubtless due in part to changes in exposure of the anemometer, and in part to the gradual erection of buildings near the Weather Bureau station. The instrument was located on the roof of Brown's Block from 1878 to 1883, where the exposure was probably uniform in all directions. In 1883 it was transferred to a position 71 feet distant from the dome of the Washburn Observatory, and in the lee of the dome for westerly winds. In 1904 the instrument was installed on the roof of North Hall. At that time, at least two buildings in the neighborhood of the station presented obstructions from the East and Southwest, respectively. Since then, extensive additions have been made to these buildings, and several new buildings have been erected close to the station. This progressive increase of the number and size of obstructions in the neighborhood of the anemometer appears to account for the apparent 1 per cent decrease of the velocity.

Moreover, if α and β represent any two of the six mass motion variables, then

$$\overline{\overline{\alpha}\,\overline{\beta}} = \overline{\alpha}\,\overline{\beta} \qquad (23)$$

and

$$\overline{\overline{\alpha}\beta'} = 0 \qquad (24)$$

Evidently

$$\overline{\frac{\partial \alpha}{\partial t}} = \frac{\partial \overline{\alpha}}{\partial t} \qquad (25)$$

Making use of (23) and (24), we have from (19) the relation

$$\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta} + \overline{\alpha'\beta'} \qquad (26)$$

and, in particular, that

$$\left.\begin{array}{l} \overline{u^2_0} = \overline{u_0}^{\,2} + \overline{(u')^2} \\ \text{----------} \\ \text{----------} \end{array}\right\} \qquad (27)$$

$$\left.\begin{array}{l} \overline{u_0 v_0} = \overline{u_0}\,\overline{v_0} + \overline{u'v'} \\ \text{----------} \\ \text{----------} \end{array}\right\} \qquad (28)$$

$$\left.\begin{array}{l} \overline{\rho u_0} = \overline{\rho}\,\overline{u_0} + \overline{\rho' u'} \\ \text{----------} \\ \text{----------} \end{array}\right\} \qquad (29)$$

General circulation.—In what follows, the term *general circulation* will be used to denote winds of order sufficiently large to satisfy condition (20).

Derivation of the equations of general circulation.—The foregoing relations supply material from which can now be deduced the equations governing general circulation.

The dynamical equations of general circulation.—Making use of the mean value properties (19) to (29), and taking the time mean of relation (13) over an interval $\mathfrak{T}$, the equation of continuity becomes

$$\frac{\partial \overline{\rho}\,\overline{u_0}}{\partial x} + \frac{\partial \overline{\rho}\,\overline{v_0}}{\partial y} + \frac{\partial \overline{\rho}\,\overline{w_0}}{\partial z} = -\Phi \qquad (30)$$

where

$$\Phi = \overline{\frac{\partial(\rho' u')}{\partial x}} + \overline{\frac{\partial(\rho' v')}{\partial y}} + \overline{\frac{\partial(\rho' w')}{\partial z}} \qquad (31)$$

The partial time derivative $\frac{\partial \overline{\rho}}{\partial t}$ disappears by virtue of relation (20).

To derive the remaining equations is not so simple. The manner in which the density is implicated in equations (14) to (16) would complicate the analysis seriously, were it not for the fact that during any $\mathfrak{T}$-interval the density deviation ρ' from the wind density $\overline{\rho}$ is unquestionably but a small per cent of the latter. Consequently we may write

$$\frac{1}{\rho} = \frac{1}{\overline{\rho}}\left(1 - \frac{\rho'}{\overline{\rho}}\right) \qquad (32)$$

This approximation gives, with (19), the relation

$$\overline{\frac{1}{\rho}\alpha} = \frac{1}{\overline{\rho}}\overline{\alpha} - \frac{1}{\overline{\rho}^{2}}\overline{\rho'\alpha'} \qquad (33)$$

The first of equations (14) can now be thrown into the form [25]

$$\frac{d\overline{u}_0}{dt}=X-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x}+\tfrac{1}{3}\epsilon\frac{\partial\overline{\theta}_0}{\partial x}+\epsilon\Delta\overline{u}+\Lambda_x+\frac{1}{\overline{\rho}^2}\Gamma_x \text{ ..... } (34)$$

where

$$\Lambda_x=-\left[\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}\right]+\tfrac{1}{3}\epsilon\frac{\partial\overline{\theta'}}{\partial x}+\epsilon\Delta\overline{u'}. \quad (35)$$

$$\Gamma_x=\overline{\rho'\frac{\partial p'}{\partial x}} \text{ .... } (36)$$

Similar expressions follow for $\frac{d\overline{v}_0}{dt}$ and $\frac{d\overline{w}_0}{dt}$.

The thermodynamical equations of general circulation.—Writing equation (15) in the form

$$\frac{p}{\rho}=HT$$

and using relation (32), we have for the characteristic equation of a wind

$$\overline{p}=\overline{\rho}H\overline{T}+\Xi \text{ .... } (37)$$

where

$$\Xi=-\frac{1}{\overline{\rho}}\overline{\rho'p'} \text{ .... } (38)$$

The mean value theorems of general circulation (19) to (29), together with (32), when applied to (16), yield the energy equation

$$\vartheta\Delta\overline{T}+\overline{\mathfrak{G}}=\overline{\rho}\,c_v\frac{d\overline{T}}{dt}+\tfrac{2}{3}\overline{\rho}\,c_v\overline{\theta}_0$$

$$+\frac{\epsilon\overline{\rho}}{J}\left[\tfrac{2}{3}\overline{\theta_0}^2-\Sigma\left(\frac{\partial\overline{v}_0}{\partial x}+\frac{\partial\overline{u}_0}{\partial y}\right)^2-2\Sigma\left(\frac{\partial\overline{u}_0}{\partial x}\right)^2\right]$$

$$+\frac{1}{\overline{\rho}}[\vartheta\Psi+\Pi]+\overline{\rho}c_v\Omega+\frac{\epsilon\overline{\rho}}{J}\theta \text{ .... } (39)$$

where

$$\Psi=\overline{\rho'\Delta T'} \text{ .... } (40)$$

$$\Omega=\overline{u'\frac{\partial T'}{\partial x}}+\overline{v'\frac{\partial T'}{\partial y}}+\overline{w'\frac{\partial T'}{\partial z}} \text{ .... } (41)$$

$$\theta=\tfrac{2}{3}\overline{\theta'^2}-\Sigma\,\overline{\left(\frac{\partial v'}{\partial x}+\frac{\partial u'}{\partial y}\right)^2}-2\Sigma\,\overline{\left(\frac{\partial u'}{\partial x}\right)^2} \text{ ...... } (42)$$

and

$$\Pi=\overline{\rho'\mathfrak{G}'} \text{ .... } (43)$$

[25] Notation:

$$\frac{d}{dt}=\overline{u}_0\frac{\partial}{\partial x}+\overline{v}_0\frac{\partial}{\partial y}+\overline{w}_0\frac{\partial}{\partial z}, \text{ see equation (20)},$$

$$\overline{\theta}_0=\frac{\partial\overline{u}_0}{\partial x}+\frac{\partial\overline{v}_0}{\partial y}+\frac{\partial\overline{w}_0}{\partial z},$$

$$\theta'=\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}, \text{ etc.}$$

The constants of mass motion.—Laboratory determinations have shown that, for air within the ordinary range of pressures and temperatures,

$$\begin{aligned}\kappa&=0.000172\\ \vartheta&=0.000056\\ c_v&=0.169\\ J&=4.18\times10^7\\ \epsilon&=0.133\end{aligned} \text{ .... } (44)^{26}$$

These determinations have been carried out for nonturbulent stream-line configurations, which should be expected to indicate the true mass motion, and not time averages of the mass motion. The substantial agreement of results obtained by different methods for a variety of apparently nonturbulent motions,[27] and the extreme divergence of results where turbulence of any order has been observed,[28] seem to establish beyond a reasonable doubt that, within ordinary ranges of pressure and temperature, (*a*) the above values pertain to the mass motion proper, and (*b*) that they are practically invariant for a given gas with respect to all mass motions experimentally investigated. According to the Kinetic Theory of Gases, they should be practically invariant with respect to all possible mass motions.

Approximations applicable to general circulation.—For sufficiently large values of $\mathfrak{T}$, it is probable that the space derivatives of the wind velocities are small. Since ϵ and ϑ are also small, and $\frac{\epsilon}{J}$ is very small indeed, the dilitation and divergence terms of the momentum equations are probably negligible compared with the gradient forces $\left(-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x}\right)$, etc., and also the conduction and dissipation terms of the energy equation, compared with the terms expressing the time rates of change of internal and dilitation energy.

The equations of general circulation.—Equations (30) to (39), therefore, after subscripts and bars have been removed, assume the approximate form:

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=-\Phi \text{ .... } (45)$$

$$\frac{du}{dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\Lambda_x+\frac{1}{\rho^2}\Gamma_x \text{ .... } (46)$$

$$\frac{dv}{dt}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\Lambda_y+\frac{1}{\rho^2}\Gamma_y \text{ .... } (47)$$

$$\frac{dw}{dt}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\Lambda_z+\frac{1}{\rho^2}\Gamma_z \text{ .... } (48)$$

$$p=\rho HT+\Xi \text{ .... } (49)$$

$$\rho c_v\frac{dT}{dt}+\tfrac{2}{3}\rho c_v\theta-\mathfrak{G}=-\left\{\frac{1}{\rho}(\vartheta\Psi+\Pi)+\rho c_v\Omega+\frac{\epsilon\rho}{J}\theta\right\} \text{ ..... } (50)$$

[26] *C. G. S* units; gram-calories.
[27] See, for example, the viscosity determinations: 62, 63, 64, 65, 66, 67, 68, 69, 96.
[28] 70, 71, 72, 6, 12.

The turbulence functions.—The functions Λ and $\frac{1}{\rho^2}\Gamma$ are evidently tractions due to transfers of momentum within the air, which impede or accelerate the wind. It will be observed that $\frac{1}{\rho^2}\Gamma$ depends directly upon the wind density and on the density and pressure deviations, while Λ depends directly upon the velocity deviations alone. Λ therefore is of the nature of a *kinetic traction*, and $\frac{1}{\rho^2}\Gamma$, of a *thermodynamic traction*. It is reasonable to suppose that these two forces which, though mutually dependent differ essentially, correspond more or less closely to the observed distinction between kinetic and convective turbulence, in which case the distinction would prove to be fundamental.

The function Φ shows that the wind motion is not in general a possible mode of mass motion; that is to say, observed wind motions may show *apparent* deviations from the law of conservation of matter; such deviations are measured by Φ.

The apparent elastic behavior of a wind may also be expected to deviate from that of a perfect gas under laboratory conditions (i. e., in nonturbulent motion). In this case, the deviation is measured by the function Ξ.

Finally, Ψ, Ω, Π and θ measure the energy transfer due to kinetic and convective turbulence in the wind.

It can not be emphasized too strongly that these functions, corresponding to a given mass motion, will in general vary greatly from one value of $\mathfrak{T}$ to another. The tractions accompanying the wind represented by figure 1, for example, unquestionably differ profoundly in character and magnitude from those accompanying the wind represented by figure 4.

Empirical determination of the turbulence functions.—As should be expected from the nature of air resistances, these turbulence functions are complex in the extreme. To attempt to ascertain wind motion deviations and deviation rates, and to find mean values of their powers and products, would not be profitable, nor is it necessary. The problem here is precisely analogous to the problem of kinetic friction between rigid bodies, which latter is a traction due to the interaction of molecules in the bearing surfaces. That kinetic friction has been successfully reckoned with and usefully applied is due to the fact that empirical formulas have been invented, making it unnecessary to ascertain the detailed motion of the molecules. To investigate the turbulence functions of the general circulation by empirical methods is therefore an immediate and fundamental problem.

The principle stated at the end of the last section finds explicit application here. The order of the wind under consideration will determine the forms to be assigned empirically to the turbulence functions, as well as the values of such numerical coefficients as they may contain. Turbulence laws governing a wind around an aëroplane strut and defined for ten-second means, would doubtless have little in common with turbulence laws governing a general circulation of the atmosphere defined for $\mathfrak{T}$-intervals of thirty years.

Geographical coordinates.—For the purpose of investigating the turbulence functions of general circulation, it is advantageous to transform the equations of motion by replacing the fixed, rectangular coordinate system which we have been using by a system of geographical coordinates moving with the earth. Assuming rectangular axes fixed instantaneously at point P as in figure 5, then

$$u = r\cos\theta\cdot(\dot{\varphi}+\varpi),\ v = r\,\dot{\theta},\ w = \dot{r} \qquad (51)^{29}$$

where θ is the North latitude of P, φ is its East longitude, and r, its distance from the earth's center. The dots denote time rates. If E is the eastward component of the relative wind velocity at P, and N and V, the respective northward and vertical components, then

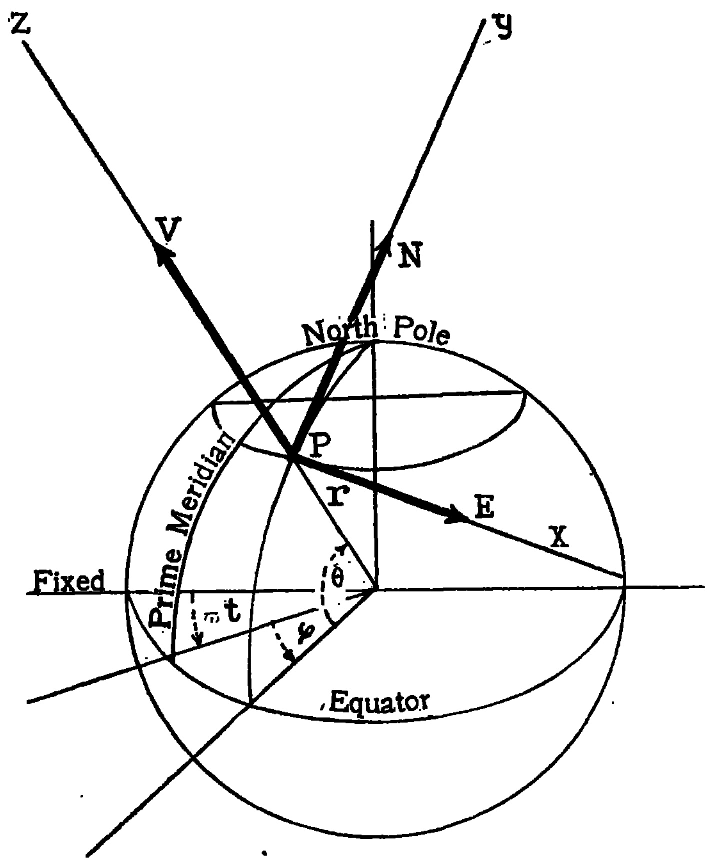

FIG. 5.—Geographical coordinates.

$$E = r\cos\theta\cdot\dot{\varphi},\ N = r\,\dot{\theta},\ V = \dot{r} \qquad (52)$$

Since the left-hand member of (45), if equated to zero, gives the equation of continuity for the detailed motion (not the average motion) of a continuous medium, the transformation of the equation of continuity is readily effected by computing the flux of such a medium into a geographical volume element. The eastward flux is

$$-\frac{1}{r\cos\theta}\frac{\partial(\rho E)}{\partial\varphi}d\tau dt,$$

the northward flux is

$$-\frac{1}{r}\frac{\partial(\rho N)}{\partial\theta}d\tau dt$$

and the upward flux is

$$-\frac{\partial(\rho V)}{\partial r}d\tau dt.$$

29 The number ϖ is the angular velocity of the earth's rotation: 0.00007292 radians per second.

The total net flux, however, would be

$$\frac{\partial \rho}{\partial t} d\tau dt;$$

consequently, the first term, plus the last three with signs reversed, give the desired function.

The momentum equations are expeditionsly transformed by the use of Lagrange's equations [30]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i - \frac{1}{\rho}\frac{\partial p}{\partial q_i} + \Lambda_i + \frac{1}{\rho^2}\Gamma_i \text{.......} (53)$$

where T is the kinetic wind energy per unit mass of air referred to the fixed, rectangular axes, and Q_i is the coefficient of δq_i in the expression for the virtual work of the external forces, that is due to a displacement δq_i during which the other two coordinates are held fast. The absolute kinetic wind energy is readily seen to be

$$T = \frac{1}{2}\left\{r^2 \cos^2 \theta \cdot (\dot{\varphi} + \varpi)^2 + r^2\dot{\theta}^2 + \dot{r}^2\right\} \text{........} (54)$$

and a simple computation based on relations (51) and (54) yields the desired equations.

The time rate of cubical dilitation, which we shall now denote by div q, where q is the vector wind velocity, is readily obtained by observing that, at any instant, u, v, and w are in general functions of r, θ and φ, and that r, θ and φ are, in turn, functions of x, y, and z, so that the variation in u due to a virtual displacement in the x-direction is given by

$$\delta u = \frac{\partial u}{\partial r}\delta r + \frac{\partial u}{\partial \theta}\delta\theta + \frac{\partial u}{\partial \varphi}\delta\varphi.$$

But the x-displacement produces only higher order changes in θ and r, while it produces a variation in φ which is given by

$$r \cos \theta \cdot \delta\varphi = \delta x;$$

whence

$$\frac{\partial u}{\partial x} = \frac{1}{r \cos \theta}\frac{\partial E}{\partial \varphi} \text{..............} (55)$$

The partial derivatives of v and w are computed in a similar manner; consequently

$$\text{div } q = \frac{1}{r \cos \theta}\frac{\partial E}{\partial \varphi} + \frac{1}{r}\frac{\partial N}{\partial \theta} + \frac{\partial V}{\partial r} \text{.........} (56)$$

The equations of general circulation in geographical coordinates.—The above equations may now be written as follows[31]:

$$\frac{1}{r \cos \theta}\frac{\partial(\rho E)}{\partial \varphi} + \frac{1}{r}\frac{\partial(\rho N)}{\partial \theta} + \frac{\partial(\rho V)}{\partial r} = -\Phi \text{.....} (57)$$

$$\frac{dE}{dt} = \left\{\frac{EN}{r}\tan\theta - \frac{VE}{r}\right\} + \left[2\varpi N \sin\theta - 2\varpi V \cos\theta\right]$$
$$- \frac{1}{r \cos \theta}\left(\frac{1}{\rho}\frac{\partial p}{\partial \varphi}\right) + \Lambda_E + \frac{1}{\rho^2}\Gamma_E \text{................} (58)$$

$$\frac{dN}{dt} = \left\{-\frac{E^2}{r}\tan\theta - \frac{NV}{r}\right\} + \left[-2\varpi E \sin\theta\right]$$
$$- \frac{1}{r}\left(\frac{1}{\rho}\frac{\partial p}{\partial \theta}\right) + \Lambda_N + \frac{1}{\rho^2}\Gamma_N \text{................} (59)$$

$$\frac{dV}{dt} = \left\{\frac{E^2 + N^2}{r}\right\} + \left[2\varpi E \cos\theta\right] - g - \frac{1}{\rho}\frac{\partial p}{\partial r} + \Lambda_V + \frac{1}{\rho^2}\Gamma_V \text{..} (60)$$

$$p = \rho H T + \Xi \text{................} (61)$$

$$\rho c_v \frac{dT}{dt} + \tfrac{2}{3}\rho c_v\left\{\frac{1}{r \cos \theta}\frac{\partial E}{\partial \varphi} + \frac{1}{r}\frac{\partial N}{\partial \theta} + \frac{\partial V}{\partial r}\right\} - \mathfrak{G}$$
$$= -\left\{\frac{1}{\rho}\left(\vartheta\Psi + \Pi\right) + \rho c_v \Omega + \frac{\epsilon\rho}{J}\theta\right\} \text{.........} (62)$$

The quadratic terms in E, N, and V occurring in the momentum equations are negligible compared with the accelerations of Coriolis,[32] provided that $\frac{E}{r}$, $\frac{N}{r}$, and $\frac{V}{r}$ are small compared with ϖ.

Viscous fluid analogies.—The classical analysis of internal stresses set up by the motion of an isotropic, homogeneous, continuous medium within the medium itself furnishes an advantageous starting point for an empirical investigation of the turbulence functions of general circulation, to which we now return. The momentum equations for a thin layer of any continuous medium enveloping the earth may be written as follows:[33]

$$\left.\begin{aligned}
J_E &= -\frac{1}{r \cos \theta}\left(\frac{1}{\rho}\frac{\partial p}{\partial \varphi}\right) + \frac{1}{\rho}\left\{\frac{1}{r \cos \theta}\frac{\partial E_e}{\partial \varphi} + \frac{1}{r}\frac{\partial E_n}{\partial \theta} + \frac{\partial E_v}{\partial r}\right\}\\
J_N &= -\frac{1}{r}\left(\frac{1}{\rho}\frac{\partial p}{\partial \theta}\right) + \frac{1}{\rho}\left\{\frac{1}{r \cos \theta}\frac{\partial N_e}{\partial \varphi} + \frac{1}{r}\frac{\partial N_n}{\partial \theta} + \frac{\partial N_v}{\partial r}\right\}\\
J_V &= -g - \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{\rho}\left\{\frac{1}{r \cos \theta}\frac{\partial V_e}{\partial \varphi} + \frac{1}{r}\frac{\partial V_n}{\partial \theta} + \frac{\partial V_v}{\partial r}\right\}
\end{aligned}\right\} (63)$$

where J is the resultant of the relative acceleration, the small accelerations discussed in the last paragraph, and the accelerations of Coriolis. If the medium is homogenous and isotropic, and if the velocities and their space derivatives are so small that their squares and products may be neglected,[34] then the tensions or compressions per unit area within the fluid are

$$\left.\begin{aligned}
E_e &= \lambda \text{ div } q + \frac{2\mu}{r \cos \theta}\frac{\partial E}{\partial \varphi}\\
N_n &= \lambda \text{ div } q + \frac{2\mu}{r}\frac{\partial N}{\partial \theta}\\
V_v &= \lambda \text{ div } q + 2\mu\frac{\partial V}{\partial r}
\end{aligned}\right\} \text{..........} (64)$$

and the shearing stresses per unit area within the fluid are

$$\left.\begin{aligned}
E_n = N_e &= \mu\left\{\frac{1}{r \cos \theta}\frac{\partial N}{\partial \varphi} + \frac{1}{r}\frac{\partial E}{\partial \theta}\right\}\\
N_v = V_n &= \mu\left\{\frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial N}{\partial r}\right\}\\
V_e = E_v &= \mu\left\{\frac{\partial E}{\partial r} + \frac{1}{r \cos \theta}\frac{\partial V}{\partial \varphi}\right\}
\end{aligned}\right\} \text{.......} (65)$$

[30] 73, p. 343. In equations (53), $i = (1, 2, 3)$, and $q_1 = r$, $q_2 = \theta$ and $q_3 = \varphi$.

[31] $$\frac{d}{dt} = \frac{E}{r \cos \theta}\frac{\partial}{\partial \varphi} + \frac{N}{r}\frac{\partial}{\partial \theta} + V\frac{\partial}{\partial r} \text{..............................} (56a)$$

When this operator is applied to E, N, or V, the result will be referred to as a relative acceleration. The absence of the partial time derivatives is due to the hypothesis expressed by equation (20), and to the definition of general circulation.

[32] The terms inclosed in square brackets. See 74.

[33] E_e, etc., are the internal stresses due to the viscosity of the medium. Each stress lies in a plane perpendicular to the direction denoted by the subscript, and the direction of the stress is denoted by the capital letter. The depth of the medium is assumed to be small compared with the radius of the earth.

The equations for the motion of a viscid, continuous medium in their classical form were first suggested by Navier (75) and subsequently derived by Saint-Venant (76) and Stokes (52 and 77). See also (29) and (78).

[34] See (73, p. 554). The numbers λ and μ are the viscosity coefficients for the medium. If $3\lambda + 2\mu = 0$, then the medium is such that uniform dilitation without shearing can take place without dissipation of energy. There seems to be no good reason for supposing this relation to hold for viscous media in general; see (76, p. 1240); (52, p. 287); (77, p. 16); (80, p. 95); (81, p. 174); (78, Vol. 1, p. 22); (82, p. 462). But the relation probably is true for gases; see (79, p. 116, 193).

From the standpoint of the Kinetic Theory of Gases, all of these stresses and hence the resulting tractions are due to molecular transfers of momentum in the medium. In the preceding analysis, we have seen that the *wind* tractions Λ and $\frac{1}{\rho^2}\,\Gamma$ are due to turbulent transfers of momentum in the wind itself. But winds are anisotropic, at least to transfers effected by convective turbulence, and the velocities and their space derivatives are by no manner of means so small that their squares and products can be neglected. Equations (64) and (65) therefore can not possibly represent the internal mechanism of a wind in a state of convective turbulence.

On the other hand, if a viscous fluid were anisotropic, but (*a*) the tensions or compressions due to the fluid motion were proportional to the differential velocities, and (*b*) the shearing stresses obeyed a similar law and acted only in horizontal planes, then relations (64) and (65) would read

$$\left.\begin{aligned} E_e &= \frac{2\mu}{r\cos\theta}\frac{\partial E}{\partial\varphi} \\ N_n &= \frac{2\mu}{r}\frac{\partial N}{\partial\theta} \\ V_v &= 2\mu\frac{\partial V}{\partial r} \end{aligned}\right\} \text{- - - - - - - - - - - - - - - - (66)}$$

$$\left.\begin{aligned} E_n &= N_e = V_e = V_n = O \\ N_v &= \mu\frac{\partial N}{\partial r} \\ E_v &= \mu\frac{\partial E}{\partial r} \end{aligned}\right\} \text{- - - - - - - - - - - - - (67)}$$

Equations (63) show that the corresponding eastward traction would be

$$\epsilon\left\{\frac{2}{r^2\cos^2\theta}\frac{\partial^2 E}{\partial\varphi^2}+\frac{\partial^2 E}{\partial r^2}\right\}$$

the northward traction would be

$$\epsilon\left\{\frac{2}{r^2}\frac{\partial^2 N}{\partial\theta^2}+\frac{\partial^2 N}{\partial r^2}\right\}$$

and the vertical traction would be

$$\epsilon\left\{2\frac{\partial^2 V}{\partial r^2}\right\}$$

where $\epsilon=\frac{\mu}{\rho}$, which is practically constant.

Finally, if the condition be imposed that the pressures are small compared with the shearing stresses, it is evident from (63) that the eastward traction becomes

$$\epsilon\frac{\partial^2 E}{\partial r^2},$$

the northward traction,

$$\epsilon\frac{\partial^2 N}{\partial r^2},$$

and the vertical traction,

$$O. \qquad \text{- - - - - - (69)}$$

Consideration of the motion [35] defined by the relations

$$E=ar,\ N=br,\ V=O,\ \rho=\text{const.}$$

will show that a medium satisfying conditions (69) is vertically isotropic to momentum transfers producing the horizontal tangential resistances (67).

The analogy of these systems of resistances may serve as an advantageous starting point for the study of wind tractions. Since the character of these tractions depends fundamentally on the $\mathfrak{T}$-interval, we shall accordingly assume, for the sake of definiteness, that the averages are computed over intervals of thirty years.

For general circulation of this order there is no good reason to suppose that the winds are anisotropic to the action of *kinetic* turbulence. The analogy suggests that internal stresses, due to turbulence of this kind, may be linear functions of the differential velocities in the manner indicated by relations (64) and (65); [36] in which case Λ_E, for example, would be a linear function of the divergence of the wind velocity q and the Laplacian of E. Such a law might reasonably be expected to hold for winds having differential velocities as high as 0.01 centimeter per second per centimeter. For higher differential velocities it might be necessary to express the stresses as homogeneous quadratic functions of the differential velocities. Higher differential velocities, however, probably do not occur in general circulation.

Convective turbulence evidently obeys a very different law. While winds may be isotropic to the action of kinetic turbulence, they are certainly anisotropic to momentum transfers due to convection. In fact, convective turbulence for the most part must act vertically across horizontal surface elements. The simplest and probably the most plausible hypothesis is therefore that suggested by relations (69), namely,

$$\left.\begin{aligned} \Gamma_E &= \frac{\mu}{\rho^2}\frac{\partial^2 E}{\partial r^2} \\ \Gamma_N &= \frac{\mu}{\rho^2}\frac{\partial^2 N}{\partial r^2} \\ \Gamma_V &= O \end{aligned}\right\} \text{- - - - - - - - - - - - - - (70)}$$

where μ is a constant depending on $\mathfrak{T}$.

The functions Ψ, Ω, Π, and θ present very much the same problem as the tractions. On account of the extreme smallness of the coefficient of θ, it is probable that kinetic turbulence can here be neglected. As to the relative values of the terms in Ψ, Ω, and Π, nothing at present can be said. A practicable working hypothesis is suggested by the analogy of Fourier's law of

[35] This motion satisfies the equation of continuity. The stream lines are

$$b\varphi = a\theta + a\log\beta\,(\tan\theta+\sec\theta)$$

where β is a parameter varying from one curve to another.

[36] *The dynamical significance of the barometer.*—If governed by the laws expressed by (64), the total normal internal pressure per unit area would be

$$\left.\begin{aligned} P_e &= -p+E_e \\ P_n &= -p+N_n \\ P_v &= -p+V_v \end{aligned}\right\} \text{. . . . . (69a).}$$

These relations throw some light on the dynamical significance of the barometer. Let us imagine an ideal shelter constructed on the principles of a lantern, allowing indirect contact between the exterior and interior air, but excluding the wind. The dimensions of this ideal shelter are microscopic and of the same order as those of the cell $d\tau$, in which we shall suppose it to be placed. This shelter, which is stationary, obstructs the wind, thus setting up extraneous disturbances, influencing to an unkonwn extent the behavior of the instruments inside. One of the latter, an ideal, microscopic barometer registering the most rapid and minute pressure fluctuations, would, however, indicate the aërostatic pressure p in the *shelter*, for the shelter would evidently eliminate the aërodynamic pressures E_e, N_n and V_v. Assuming that the turbulence set up by the shelter on its exterior, together with the destruction of the mass motion in its interior, does not result in totally transforming the character of the molecular motion normally occurring in the geometrical *cell* $d\tau$, the ideal instrument would yield an approximate value for the pressure p of the classical Kinetic Theory of Gases, as defined by equation (11). Time averages of successive readings made at sufficiently near intervals would therefore give the aërostatic wind pressure $\bar{p}$ as defined by equation (17).

In the actual case the analogon of the microscopic shelter is the closed room in which the standard barometer is suspended. The latter does not respond to rapid and minute pressure fluctuations (2), and consequently it yields at a single reading an approximate time mean corresponding to an indeterminate time interval. When, however, successive readings of the standard instrument under standard conditions are averaged over a fixed and definite interval $\mathfrak{T}$, the resulting means apparently afford a more or less rough approximation to the true aërostatic pressure $\bar{p}$ of the wind. (See equation (17)). How nearly this approximation represents the true value is a question which would well repay experimental investigation.

heat conduction in a continuous, homogeneous medium, namely, that convective energy transfer through a thin wind stratum is directly proportional to the temperature difference of its boundaries and to the time, and inversely proportional to the thickness of the stratum. If to this is added the assumption that transfers occur in the vertical direction only, the corresponding turbulence function will be proportional to

$$\frac{1}{\rho}\frac{\partial^2 T}{\partial r^2} \qquad (71)$$

or else to

$$\rho\frac{\partial^2 T}{\partial r^2} \qquad (72)$$

The functions Φ and Ξ occurring respectively in the equation of continuity and in the characteristic equation are at present entirely problematic, but it is probable that, as the anology of viscous motion suggests, these functions may play an unimportant rôle.

Studies of resistance functions.—The science of the internal mechanism of winds is a new one. It opens a wide field for research, the results of which can not fail to prove of the utmost interest and utility. Attempts have already been made to devise resistance functions and to evaluate the corresponding constants. All of these attempts have been made under the assumption of a general air drift differing from the detailed fluid motion; but in no case has the exact nature of the drift under consideration been defined, nor has the all-important question of wind order been investigated.

Guldberg and Mohn [37] assumed that the internal resistance F per unit mass of air was proportional to W, the horizontal wind velocity:

$$\left.\begin{array}{l} F=\mu W \\ W=\sqrt{E^2+N^2} \end{array}\right\} \qquad (73)$$

and that the resistance acted directly against the wind. Neglecting relative accelerations and the small quadratic terms in E, N, and V, the two hydrodynamical equations corresponding to (58) and (59) yield the relation

$$\frac{1}{\rho}\frac{\partial p}{\partial s}=2\varpi\, W\sin\theta-F_s$$

where s is the direction in which the pressure gradient is steepest. Since the "deflective force" $2\varpi W\sin\theta$ acts normal to the path and, in the Northern Hemisphere, to the right, it follows that [38]

$$\left.\begin{array}{l} -\dfrac{1}{\rho}\dfrac{\partial p}{\partial s}\cos\alpha=\mu\, W \\ -\dfrac{1}{\rho}\dfrac{\partial p}{\partial s}\sin\alpha=2\varpi\, W\sin\theta \end{array}\right\} \qquad (74)$$

Consequently

$$\mu=(2\varpi\sin\theta)\,\mathrm{cotan}\,\alpha.$$

Values of the deflection angle obtained from weather maps gave

$$0.0002>\mu>0.00008.$$

To check the resistance law on which these results depended, advantage was taken of the fact that sufficient data were at hand to over-determine equations (74). The ratio of W to $\frac{\partial p}{\partial s}$ was computed from previously obtained values of μ, and, as was to have been expected from the nature of the resistance function, the results did not check well with observations.

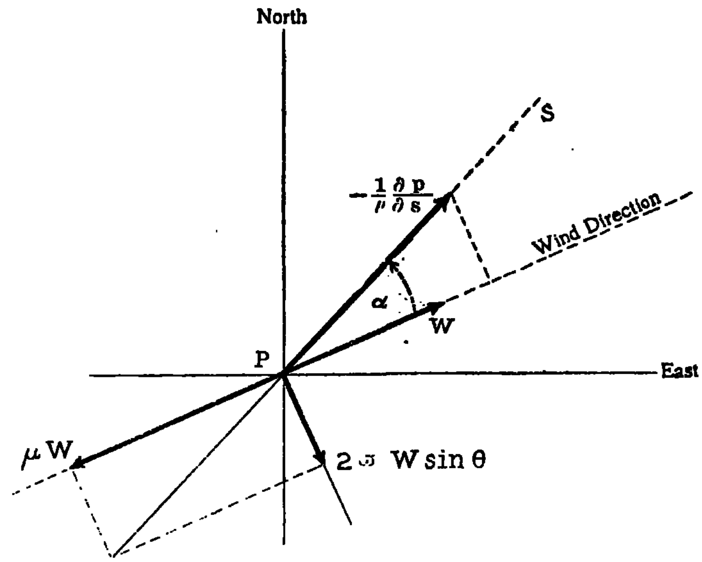

FIG. 6.—Guldberg and Mohn deflection angle, α. The direction s is that in which the gradient force is at a maximum.

Oberbeck [39] assumed that resistances obeyed the law

$$F=\frac{\mu}{\rho}\Delta q,$$

where μ was a constant depending on the motion.

Åkerblom,[40] in his researches relating to the winds over Paris, assumed that

$$F_E=\frac{\mu}{\rho}\frac{\partial^2 E}{\partial r^2},\quad F_N=\frac{\mu}{\rho}\frac{\partial^2 N}{\partial r^2}$$

where, as with Oberbeck, μ was a constant depending on the internal irregularities of the average drift. Neglecting accelerations, and imposing restrictions on the velocities, he found that

$$\mu=95\frac{\text{gm.}}{\text{cm. sec.}}$$

Hesselberg and Sverdrop [41] assumed that

$$F=\kappa\frac{W_{500}-W_0}{500}-k\,W_0.$$

The first term was introduced under the supposed necessity of representing separately the traction of the upper layers upon the lower ones. The second term was intended to represent the tractions at the ground. The difference quotient

$$\frac{W_{500}-W_0}{500}$$

was evidently a rough approximation not to $\frac{\partial^2 W}{\partial r^2}$ but to $\frac{\partial W}{\partial r}$, and as here used was equivalent to the assumption of horziontal shearing stresses in the upper air which are proportional not to the differential velocities but to the velocities themselves. It is difficult to see how such a

[37] (70. See also 83.) Stokes pointed out in 1845 that "The main part of the resistance of fluids depends on the formation of eddies." 53, p. 99.

[38] See fig. 6. The angle α was called the *deflection angle*.

[39] 84. See also the memoirs of Boussinesq, 85 and 86.

[40] 71. See also 87.

[41] 72. See also 88.

law could prove valid. Possibly with this in mind, Hesselberg and Sverdrop carried out another determination depending on the law already suggested by Åkerblom. In this case the second derivatives were obtained from observations.

A study of a very restricted type of eddy motion superimposed upon a horizontal air drift led Taylor [42] to adopt for horizontal tractions the law assumed by Åkerblom. Similar considerations led to the assumption of a law to the effect that the transfer of heat by turbulent motion is proportional to

$$\frac{\partial^2 \theta}{\partial z^2},$$

where θ is the potential temperature of the air at altitude z. This law would necessitate either a horizontal downward transfer of energy by turbulence in the isothermal region, where the potential temperature increases with altitude, or else it would necessitate a strong variation with altitude of the proportionality factor. Since observations indicate distinctly a state of nonturbulence in the isothermal region,[43] the former alternative is untenable. The latter alternative imposes upon the proportionality factor the burden of an entirely unknown function, and therefore yields no additional information as to the general circulation.

These various studies seem to be significant for the problem of general circulation (*a*) in that they contain proposed resistance laws which may prove useful as working hypotheses, and (*b*) in that they illustrate the importance of a fundamental principle which, curiously enough, seems to have been entirely overlooked except by Guldberg and Mohn, namely, that empirical determinations of resistance constants are meaningless unless sufficient data are brought to bear to over-determine the equations of motion, and so *to verify the empirical resistance laws on which the results depend*. It should be added that empirically determined results should be expected to hold only for *definite vaules of $\mathfrak{T}$ corresponding to the order of the winds under investigation.*

[42] 94, 95, 99.

[43] Above the substratosphere. See 58 and 59.

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